Frequency Response Analysis

Technical Report 10

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1.   Introduction

Frequency response analysis is the technique whereby a sinusoidal test signal is used to measure points on the frequency response of a transfer function or impedance function. The basic set-up is shown in Figure 1 in which a sine wave $u(t)$ is applied to a system with transfer function $G(s)$. After transients due to initial conditions have decayed away, the output $y(t)$ becomes a sine wave but with a different magnitude Y and relative phase Φ. The magnitude and phase of the output $y(t)$ are in fact related to the transfer function $G(s)$ at the frequency (ω rad/s) of the input sinusoid.

$$u = U \sin \omega t \qquad \boxed{G(s)} \qquad y = Y \sin(\omega t + \Phi)$$
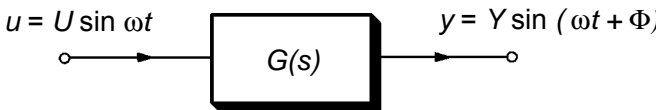
Fig. 1   *Linear transfer function with sinewave input.*

Thus $\dfrac{Y}{U} = \left| G(j\omega) \right| = $ gain at ω ....

1.

and $\Phi = \angle G(j\omega) = $ phase at ω ....
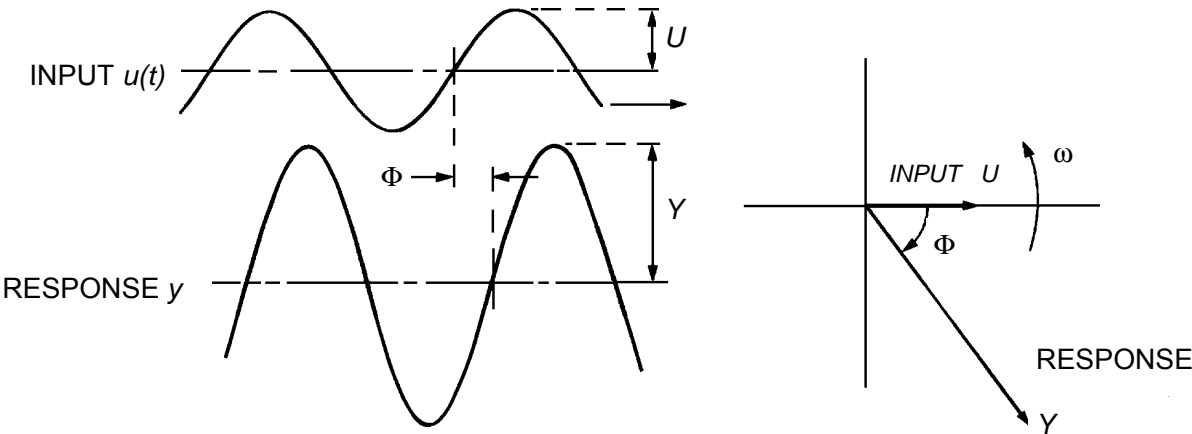
Fig. 2   *Sinewave input and steady-state sinewave response as displayed on an oscilloscope. The corresponding vector notation is shown on the right.*

The relative phase and magnitude of the input and the output waveforms can be directly measured from an oscilloscope trace (Figure 2), although this is inherently of poor accuracy and in practice more sophisticated methods are required to remove certain errors which occur due to noise, non-linearity and the like.

The great advantage of frequency response analysis lies in its frequency selective nature. Just one component of the frequency spectrum is extracted, and the corresponding system response at that frequency can be measured with great precision. This has significant advantages where the system under consideration has resonant features. Specifically, the various resonances can be excited and studies separately without disturbing other oscillatory modes.

Fig. 3a

locus of G (j ω)
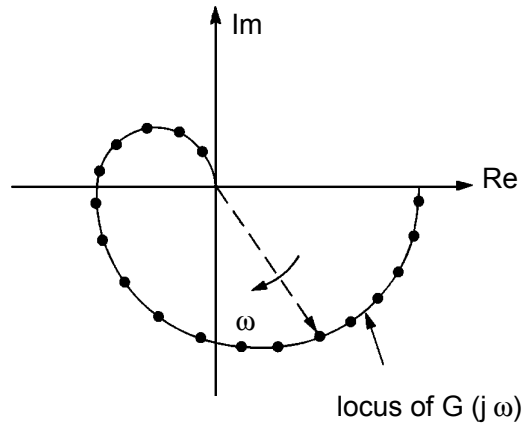
Fig. 3b   *The frequency axis is often plotted to a log scale*
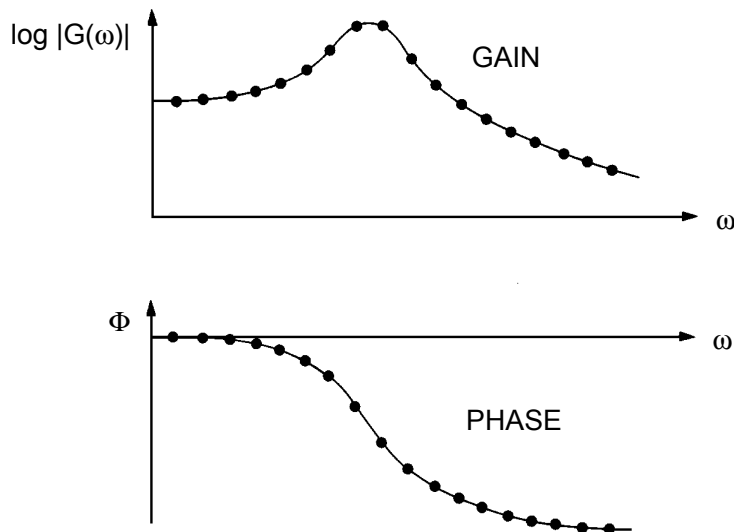
Fig. 3   *Presentation of frequency response information*

By sequentially measuring the gain and phase at various frequencies, a picture of the system frequency response can be built-up and plotted either as a diagram in the complex plane (Fig. 3a) or a joint graph of gain against frequency (Figure 3b). The latter are often termed Bode diagrams because of their use in Bode's control system design methods.

In addition to its frequency selective nature frequency response analysis has the advantage that since waves have well-defined amplitude characteristics. In particular the maximum amplitude of a sine wave is only √2 times greater than the root mean square value. The ratio

$$CF = \frac{\text{peak value of signal}}{\text{root mean square value of signal}}$$

is termed the crest factor (CF) and is an important factor is designing test signals for systems analysis. Specifically, one usually wishes to inject as much power (as determined by the mean square signal level) into a system as compatible with a linear system response.

System linearity is usually related to the peak test signal excursion which can be tolerated before non-linear effects become evident. The "best" excitation in this respect is the binary waveform with crest factor of infinity. With a crest factor of 1.41 the sine wave is reasonably close to the theoretical best, while retaining the great merit of frequency selective system excitation.

2.   Basic result

Here we present a simple demonstration of the result, quoted earlier, that the steady state output of a linear system excited by a sinewave is itself a sine wave with gain and phase related to the frequency response as per equation 1.

The transform output of the system in Figure 1 can be written as

$$Y(s) = G(s)\ U(s)$$ ……………………………………………………………… **2.**

where $U(s) = \mathsf{L}[U\sin\omega t] = \dfrac{U\omega}{\left(s^2 + \omega^2\right)}$ ……………………………………………... **3.**

Suppose the system $G(s)$ is stable with distinct poles p_i [i = 1, - - - - n]   then the equation 2 can be written as

$$Y(s) = U\left[\sum_{i=1}^{n} \frac{A_i}{(s+p_i)} + \frac{B_1}{(s-j\omega)} + \frac{B_1{}^*}{(s+j\omega)}\right]$$ ………………………………… **4.**

with corresponding inverse Laplace transform

$$y(t) = U\left[\sum_{i=1}^{n} A_i \exp[-p_i t] + 2\,|B_1|\,\cos(\omega t + \angle B_1)\right]$$ ………………………………… **5.**

where $|B_1|$ is the magnitude of B_1
            $\angle B_1$ is the angle or argument of B_1

and, by Heavisides rule, $B_1 = \dfrac{G(j\omega)}{2j}$ ……………………………………………………… **6.**

Now, after a sufficient time to allow the transients to decay, the "steady state" output is obtained. Combining equations 5 and 6 this is

$$y(t) = U\,|G\,(j\omega)|\,\sin\,(\,\omega t + \angle G(j\omega)\,)$$ ………………………………………… **7.**

3. Implementation

The direct relative gain and phase measurement procedure mentioned earlier has certain practical draw-backs as a means of measuring Frequency Response. These relate to the corruption or distortion of output signals as they occur in engineering systems. Signal corruption is due to extraneous noise which infects the output measurements (as in Figure 4a) and makes it impractical to accurately measure gain and phase. Signal distortion arises from non-linear effects in a system, or measurement procedure, such that the measured output is no longer sinusoidal (as in Figure 4b) and again direct measurement of gain and phase is difficult.
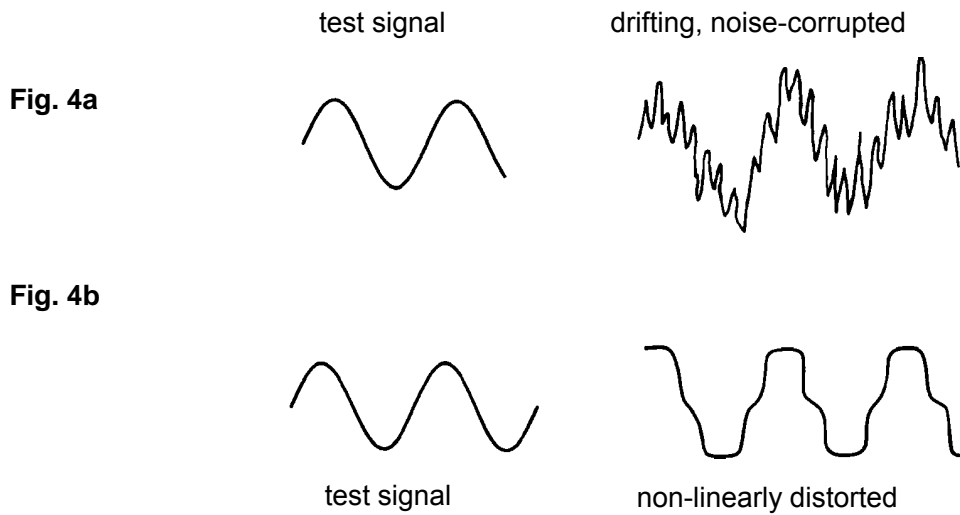
test signal   drifting, noise-corrupted

Fig. 4a

Fig. 4b

test signal   non-linearly distorted

Fig. 4   *In practical systems, although the applied test signal may be sinusoidal, the response is often corrupted due to noise, drift or non-linearity in the system being tested*

Both the problems of non-linear distortion and noise corruption are overcome in the measurement scheme of Figure 5, in which the measured output $y(t)$ is first multiplied by sine and cosine respectively and then integrated. Notice the twin processing of multiplication and averaging which are present in all noise suppressing identification and estimation techniques.

Effectively the set-up of Figure 5 correlates the output $y(t)$ with different versions of the input, hence the term correlation frequency response analyser.

Briefly, the correlation analyser functions as follows. The measured system output is multiplied by both the sine and cosine of the test frequency ω. The results of the multiplications are then fed to two identical integrators, where they are averaged over T seconds. As the averaging time increases the contribution of all unwanted frequency components in $y(t)$ go to zero, and the integrator outputs $R(T), I(T)$ become constant values which depend only on the gain and phase of the system transfer function at the test frequency.

In practice, the averaging is conducted over a finite time interval T, and it is necessary, for reasons to be explained later, that T be made an integer multiple of the test frequency period.

In mathematical terms, the correlation analyser functions as follows:

$$u(t) = U \sin \omega t$$
$$y(t) = Y \sin (\omega t + \Phi)$$

where $\quad Y = U \ |G (j\omega)| \ $ and $\ \Phi = \angle G(j\omega)$

Then the sine channel output $R(T)$ is given by

$$R(T) = \frac{U}{T} \ |G(j\omega)| \ \int_{0}^{T} \sin \omega t \sin(\omega t + \Phi) dt \quad \text{...} \quad \textbf{8.}$$

$$= \ \frac{U}{T} \ |G(j\omega)| \ \left[\cos \Phi \left(\frac{T}{2} + \frac{\sin 2\omega T}{4\omega} \right) - \sin \Phi \left(\frac{\cos 2\omega T}{4\omega} - \frac{1}{4\omega} \right) \right]$$

So that when $T = \dfrac{N\pi}{\omega}$, $N = 1,2,3$ - - - - the sine channel output is

$$R\left(\frac{N\pi}{\omega} \right) = \ \frac{U}{2} \ |G(j\omega)| \ \cos \Phi \quad \text{...} \quad \textbf{9.}$$

Similarly the cosine channel output $I(T)$ is given by

$$I(T) = \ \frac{U}{T} \ |G(j\omega)| \ \int_{0}^{T} \cos \omega t \sin (\omega t + \Phi) \ dt \quad \text{......................................} \quad \textbf{10.}$$

which can be evaluated for $T = \dfrac{N\pi}{\omega}$ as

$$I\left(\frac{N\pi}{\omega} \right) = \frac{U}{2} \ |G(j\omega)| \ \sin \Phi \quad \text{...} \quad \textbf{11.}$$
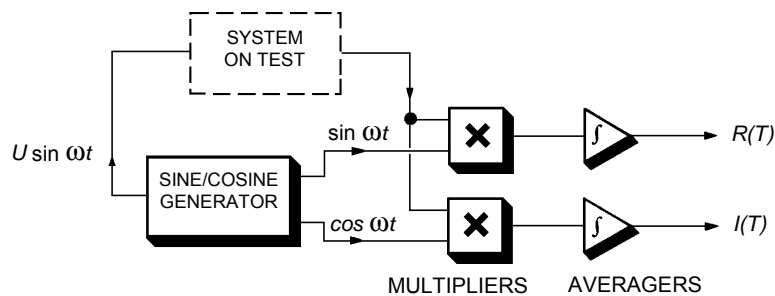

Fig. 5 *Implementation of a correlation*

Thus the frequency response function can be recovered from the signals $R(T), I(T)$ evaluated at appropriate multiples of π/ω. In practice to avoid non-linear harmonic distortion of the results, the averaging process is performed of whole cycles of the input sine wave, so that the average time T is selected as,

$$T = \ N\frac{2\pi}{\omega}, \ N = 1,2,3 \text{ - - -}$$

Note that the averaging time is given as multiples of $2\pi/\omega$. This is because averaging over a complete cycle avoids certain measurement errors associated with offsets on the system output.

4. Noise suppression

The key feature of the correlation frequency response analyser is that by increasing the averaging time T, the influence of extraneous noise on the measurement process can be reduced to very low levels indeed. To see how this works consider Figure 6 in which the true system output is corrupted by additive noise $n(t)$. Because the noise is additive the sine and cosine channels at analysis frequency ω, will each be in error an amount $\Delta R(T)$ and $\Delta I(T)$ given by

$$\Delta R(T) = \frac{1}{T}\int_0^T n(t)\sin \omega_1 t \, dt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

12.

$$\Delta I(T) = \frac{1}{T}\int_0^T n(t)\cos \omega_1 t \, dt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Now consider the sine channel error only (since the cosine error behaves in a similar manner) and interpret the action of the averaging process as a filtering operation.
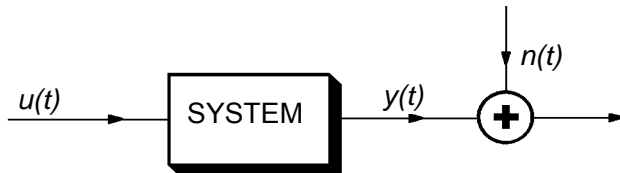

Fig. 6   *System corrupted by noise*

To do this notice that in equation 12, $\Delta R(T)$ can be written as

$$\Delta R(T) = \frac{1}{T}\int n(t)\sin\left[\omega_1(T-t)\right] dt \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \textbf{13.}$$

Because $\sin \omega_1 t = -\sin\left[\omega_1(T-t)\right]$, when $T = \dfrac{N2\pi}{\omega_1}$ $\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

Equation 13 is a convolution between the disturbance $n(t)$ and the filter with impulse response (Figure7)

$$h(\tau) = \frac{1}{T}\sin \omega_1\tau \quad 0 < \tau < T \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \textbf{14.}$$

$$= 0 \text{ elsewhere} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The frequency response of this filter is obtained by noting that it consists of the product of the function $I(\tau)$ defined by

$$I(\tau) = \frac{1}{T} \quad\quad 0 < \tau < T \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \textbf{15.}$$

$$= 0 \text{ elsewhere} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

and the sine wave $\sin \omega_1\tau$. Now the sinewave can be written as

$$\sin \omega_1\tau = \frac{1}{2j}\left(\exp[j\omega_1\tau] - \exp[-j\omega_1\tau]\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \textbf{16.}$$

Hence by the shift theorem in the frequency domain

$$H(j\omega) = \frac{1}{2j}\left[L(j\omega + j\omega_1) - L(j\omega - j\omega_1)\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{17.}$$

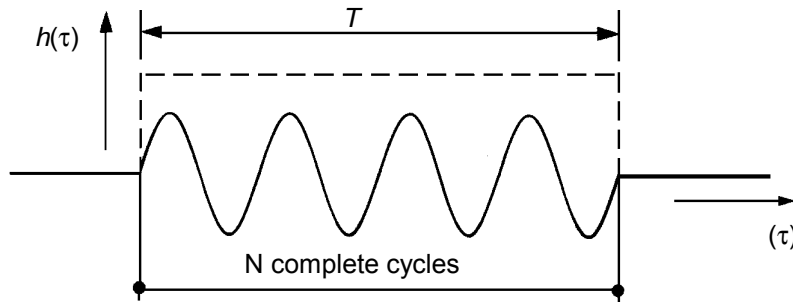$$\text{where } L(j\omega) = \int_{-\infty}^{+\infty} I(t)\exp[-j\omega t]\,dt = \frac{1}{T}\int_{0}^{T}\exp[-j\omega t]\,dt \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Fig. 7 *Impulse response of the averaging filter in the analyser.*

$$\text{hence } L(j\omega) = \frac{1}{j\omega T}\left(1 - \exp[-j\omega T]\right) = \frac{\exp\left[-j\dfrac{\omega T}{2}\right]\sin\dfrac{\omega T}{2}}{\dfrac{\omega T}{2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{18.}$$

Combining 17 and 18, the filter transfer function for $T = \dfrac{2\pi N}{\omega_1}$ is

$$H(j\omega) = \frac{\omega_1\left(1 - \exp[-j\omega T]\right)}{T\left(\omega^2 - \omega^2_1\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{19.}$$
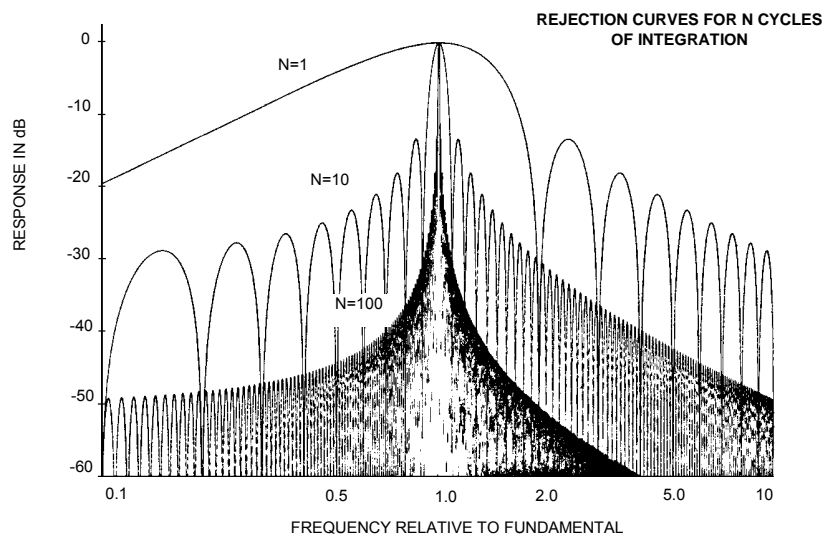
Fig. 8 *Frequency response of the averaging filter in the analyser.*

The gain function of H(jω) is plotted in Figure 8. As can be seen from the plots, the averaging associated with the correlating frequency response analyser acts as a band pass filter with centre frequency ω_1. As the average time T increases the bandwidth of the filter becomes narrower. Thus the corrupting influence of wide band noise $n(t)$ is increasingly filtered out as the correlation time is increased. This feature of increasing averaging to reduce uncertainty in measurements is a recurrent feature of dynamic measurement techniques and should be borne in mind when considering more advanced procedures which are not amenable to the simple filtering arguments used here.

5. Non linearities

The aim of frequency response analysis is to characterise the transfer function of an underlying linear dynarnic system. However, in reality few systems are linear, and so we are really measuring some linear approximation. In dynamic analysis the most troublesome forms of non-linear behaviour are associated with activating (that is, getting signals into a system) and sensing (that is, measuring a systems response). Often these forms of non-linear behaviour are severe, and may have nothing to do with the system dynamics - only how we actuate them* (Figure 9). Typically, actuation and measurement non-linearities contain no dynamics, but constitute a static gain characteristic which distorts the input / output information. The first thing the analyst tries to do is directly overcome this form of non-linearity. This is often possible by putting local feedback loops around non-linear actuators or using the methods outlined in the next section. Where this is not possible, intelligent instrumentation can sometimes compensate for non-linearity by applying an inverse non-linear operation. Many commercial data logging instruments include this feature.

When we can't get round the non-linearity, it must be accommodated in the analysis procedure. To see how this might be done consider Figure 10, which shows a non-linear characteristic which might be associated with an electric drive motor. The effect of feeding sine-waves through this non-linearity is also illustrated.

The response of any such non-linearity to a sine wave can be expressed as a sum of sine and cosine waves at the frequencies ω_1, $2\omega_1$, $3\omega_1$, $4\omega_1$, - - -. However, recall that the filter H(jω) associated with the correlation frequency response analyser has zero gain at the frequencies $2\omega_1$, $3\omega_1$, - - -, since these correspond to zeroes of L(jω) when $T = N2\pi/\omega_1$. Thus the frequency response analyser rejects all the harmonic components of output $x(t)$ (Figure 11) generated by non-linear distortion, and effectively measures the fundamental component of response only. This has uses in control system design where the describing function method hinges upon measurement of the fundamental component of a sinusoidal frequency response. By correlating the output of a system with harmonic frequencies $2\omega_1$, $3\omega_1$, - - -, $i\omega_1$, - - -, the harmonic components of frequency response are obtainable. These higher frequency components are useful in computing the exact frequency response and qualifying the level of non-linear distortion. These harmonic measurements can also be used for exact describing function measurements.

* This is not always true, since actuators and sensors often form an integral part of a system.
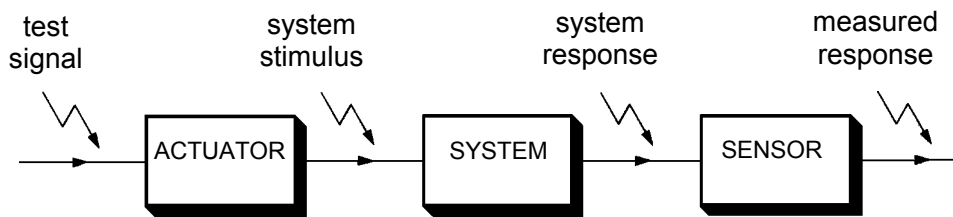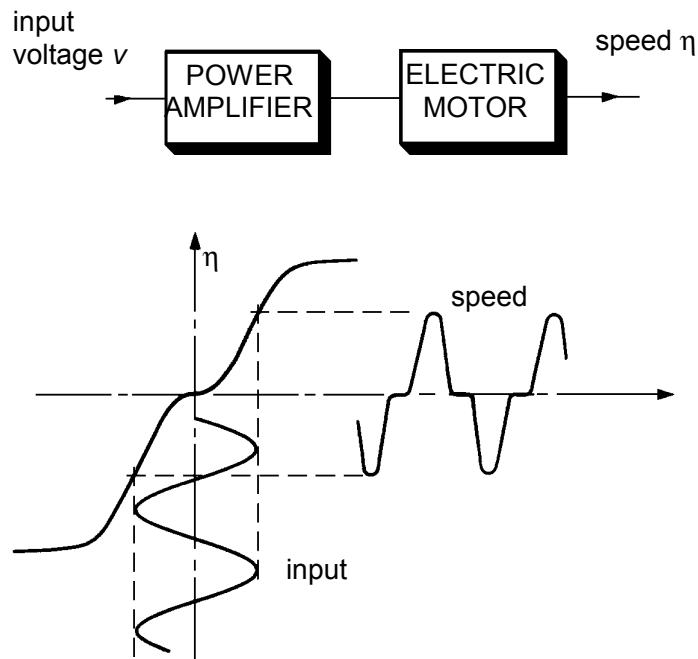
Fig. 9   *System actuation and sensing*

Fig.10   *Typical actuator non-linearity and system response.*

$$u(t) = U \sin \omega_1 t \longrightarrow \boxed{\text{NON-LINEARITY}} \longrightarrow x(t) = \sum_{i=1}^{\infty} X_i \sin(i\omega_1 t + \Phi_i)$$

RESPONSE

Envelope of FRA filter
averaging over one complete cycle

X_1 X_2 X_3 X_4 X_5 X_6

FREQUENCY

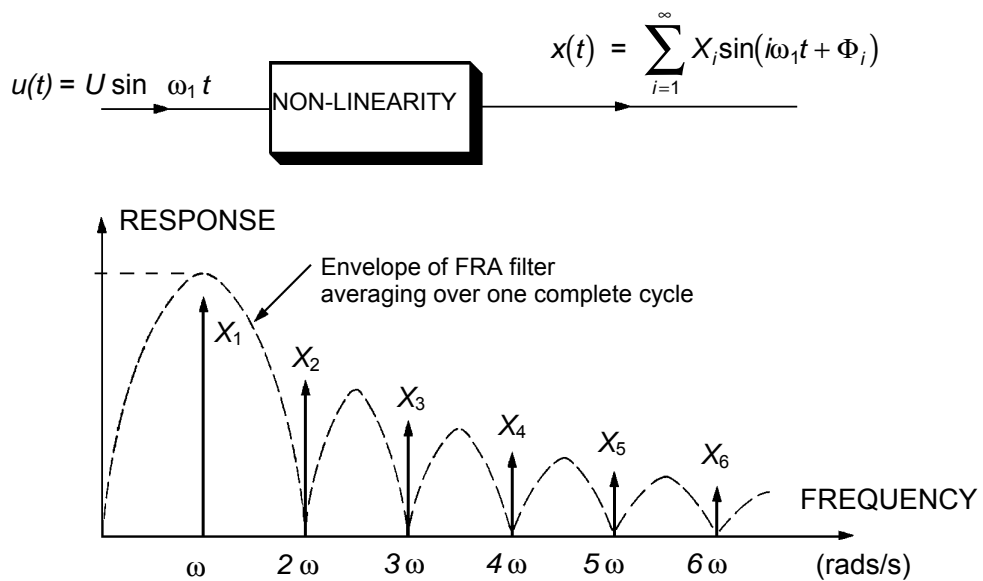ω 2ω 3ω 4ω 5ω 6ω (rads/s)

Fig. 11   *The harmonics which are present in the response from a non-linear system are suppressed by the filtering action of the analyser.*

6. Closed loop and remote input measurement

Physical constraints often make the measurement scheme of Figure 1. impractical. In particular the system to be tested might be an integral part of a larger system, or an unstable process which must be maintained under closed loop control at all times. Most modern frequency response analysers are modified to cater for such situations. Consider the two basic cases shown in Figure 12 where the system $G(j\omega)$ is either in closed loop or is down stream in a cascade of transfer functions. It is assumed that the signals $x(t),y(t)$ either side of the test system can be measured and that a test sinusoid $u(t)$ can be injected at some point which casually influences $x(t)$ and $y(t)$. The modified transfer function layout of Figure 13 is then used. In this scheme the frequency responses from $r(t)$ to $u(t)$ and $r(t)$ to $y(t)$ are measured and the frequency response of $G(j\omega)$ obtained by dividing the gain measurements and subtracting the phase measurements. The system of referencing the measurements $u(t), y(t)$ back to the test sinusoid is most important. Theoretically it relates to the method of "Instrumental variables" which grew out of econometric theory to deal with similar measurement problems in fiscal and economic data. Practically, it enables us to test closed loop systems without breaking the loop, and at the same time avoiding the ambiguity of feed-back information. The nature of this ambiguity can be seen from consideration of Figure 14. Two relations can be written for y, thus

$$y(s) = G(s)\,x(s) + n(s) \qquad \text{forward path} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y(s) = \frac{1}{F(s)}\left[-x(s) + u(s)\right] \quad \text{feedback path} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

20.

If the forward path noise $n(t)$ is zero, the forward path relation is exact and a frequency response analysis from $x(t)$ to $y(t)$ gives $G(j\omega)$.
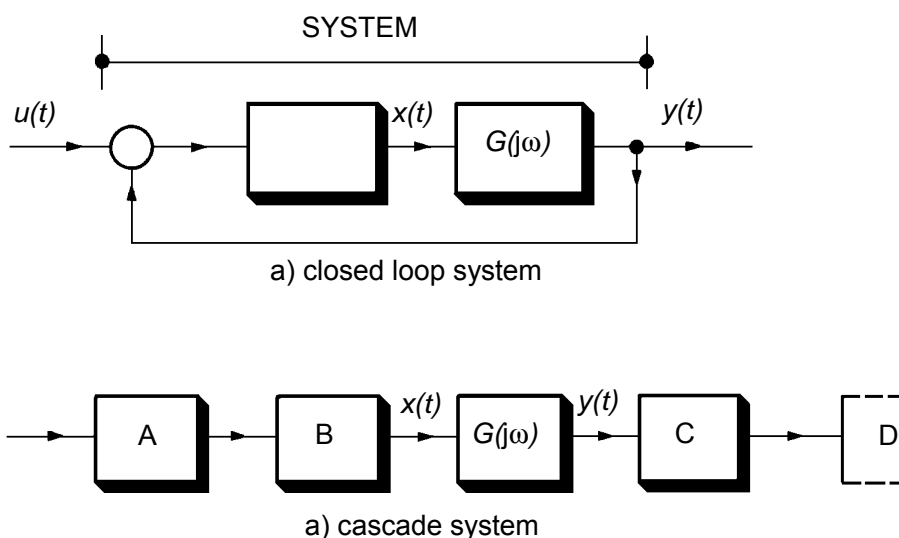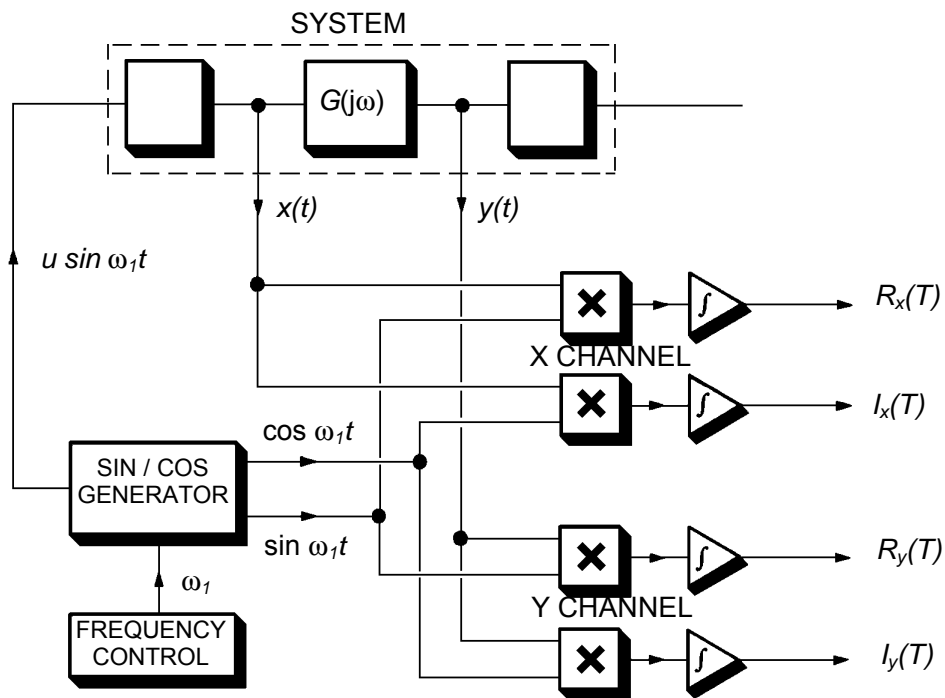

a) closed loop system

a) cascade system

Fig. 12   *Point to point frequency response analyser.*

However, if *u(t)* is very small compared with *n(t)* then the feedback path relation is exact and a frequency response analysis from *x(t)* to *y(t)* gives -1 / *F*(jω).

For intermediate values of *u(t)* and *n(t)* an estimate is obtained which is a function of *G, F* and the relative power in *u(t)* and *n(t).* The procedure of correlating back to *u(t),* as implemented in Figure 13, ensures that no ambiguity occurs. Specifically, the modified implementation measures the ratio of the transfer functions

$\dfrac{y(s)}{u(s)}$ and $\dfrac{x(s)}{u(s)}$ thus

$$\frac{\dfrac{y(s)}{u(s)}}{\dfrac{x(s)}{u(s)}} = \frac{\dfrac{G(s)}{1+FG(s)}}{\dfrac{1}{1+FG(s)}} = G(s)$$

$$\text{System frequency response at } \omega_1 = G(j\omega_1) = \frac{Ry(T)+jIy(T)}{Rx(T)+jIx(T)}$$

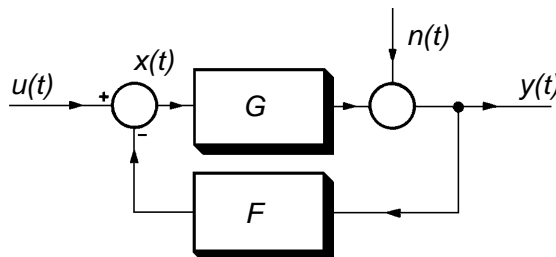Fig. 13   *A two channel point-to-point analyser.*

Fig. 14   *A closed loop system.*

It is important to realize that the feedback between two signals need not be by an explicit closed loop. The feedback may be implied by the nature of the system, as in the electrical network of Figure 15, where the two variables v_1, v_2 are not related by a simple casual transfer function, but are each joint functions of the source voltage v_0.
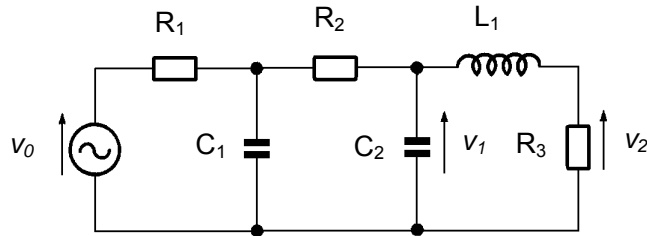

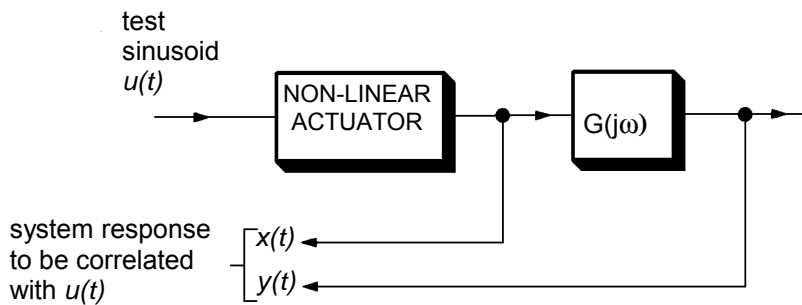**Fig. 15**   *Voltage variables in a circuit.*

Fig. 16   *Linear system with non-linear actuator.*

The modified frequency response analyser can also avoid problems with non-linear distortion. Say that block B in Figure 12 is non-linear, then by correlating both *x* and *y* back to *u(t)*, only the influence of the fundamental component of *x* is used and so *G*(jω) is correctly measured. Thus in an open loop control system with actuator non-linearity then the test scheme of Figure 16 will avoid the effects of non-linear distortion on the measurement, provided a measurement of *x(t)* can be made.

7.   Drift and offsets

In most practical situations the output of a system with a sine-wave applied to the input will consist of a normal output level, with the sinusoidal component of response superimposed. Typically (Figure 17) the output level will consist of a dc offset, combined with a low frequency drift. Moreover, the amplitude of offset and drift components maybe much larger than the test signal component. It is therefore important to find ways of removing low frequency components from the system output to prevent them distorting the measurement process.
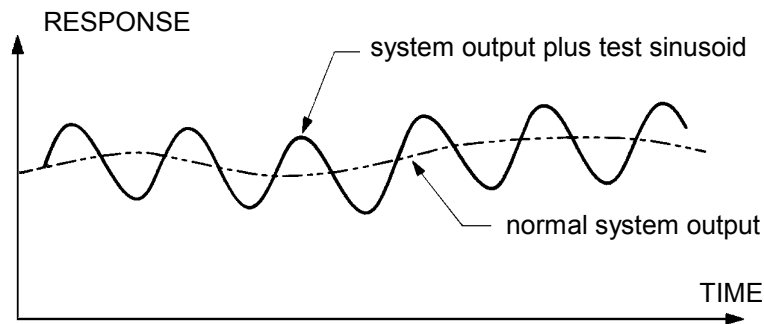

Fig. 17   *Measured system output showing normal operating level*
            (comprising drift, offset etc.) and response to a test sinusoid.

The most direct practical procedure is to use ac coupling with a capacitor. This provides a high pass filter which removes low-frequency components of response but leaves the sine-wave component uncorrupted. However, it is interesting to notice that the frequency response averaging filter $H(j\omega)$ has a zero at $\omega = 0$, so that constant components of output are always exactly cancelled out. Moreover, as the averaging number N is increased (Figure 8), the filtering out of low frequency drift becomes more effective. In practical situations the primary method of removing offsets and low frequency drift is by high-pass filtering or subtracting an estimate of the offset from the original signal. This removal at source of the major part of the offset drift improves the dynamic resolution of the frequency response measurement, and allows the natural filtering action of $H(j\omega)$ to suppress residual effects.

8. Applications

These brief notes outline some typical applications of the frequency response analyser.

(i) **Harmonic Analysis**

In practice one often meets engineering signals which are periodic with a well defined (i.e. noise-free) shape. In such cases it is frequently useful to analyse the harmonic structure of these waveforms, since the occurrence of certain harmonics may give insight into their source. By the same token there may be a requirement that certain harmonic components must be less than a pre-specified level. This is true in, for example, testing commercial signal generators where part of the technical specification refers to permitted harmonic levels.
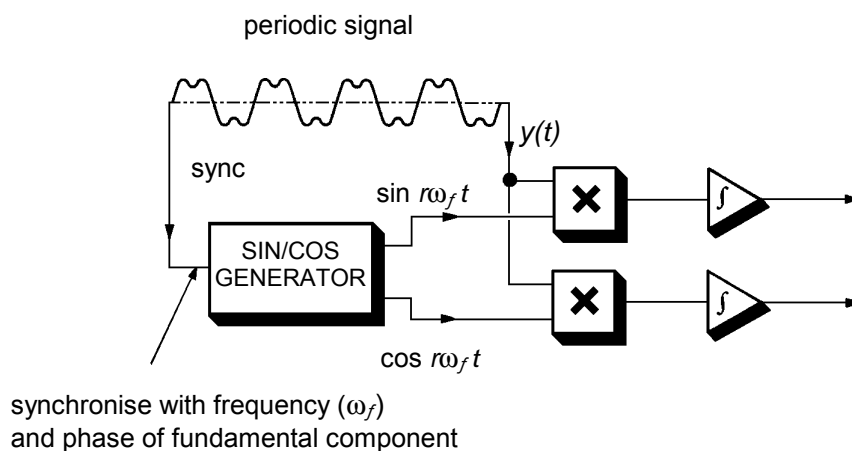

Fig. 18 *The frequency response analyser used as a harmonic analyser.*

The frequency response analyser can readily be adapted as a harmonic analyser by adding, to the circuit of Figure 5, the facility to synchronise the frequency ω_f and phase of the test sinusoid with the fundamental period of the periodic signal. The periodic signal is then treated as if it were the system output *y(t)* (see Figure 18) and correlated with the harmonic frequencies $r\omega_f$ (r=1, 2, 3, - - -) generated in the frequency response analyser. The sine and cosine output channels then give the relative gain and phase of the harmonic components. The use of harmonic frequencies in this way can also be applied to measure the harmonic content of response of a non-linear system.

ii)   Transfer Function Analysis of AC Systems

In addition to conventional dc feedback control systems there is a class of servo-mechanisms which employ amplitude modulated ac signals. The frequency response analyser is particularly effective at response testing this kind of control system for three reasons:

a)   It is particularly easy to modulate and demodulate the amplitude of an ac signal with sinewaves.

b)   The low crest factor of a sinewave test signal means that a relatively high level of modulation can be achieved before non-linear affects occur.

c)   If the demodulated signal still contains significant components of the carrier frequency, the filtering action of the correlation frequency response analyser will tend to suppress the carrier frequency components.
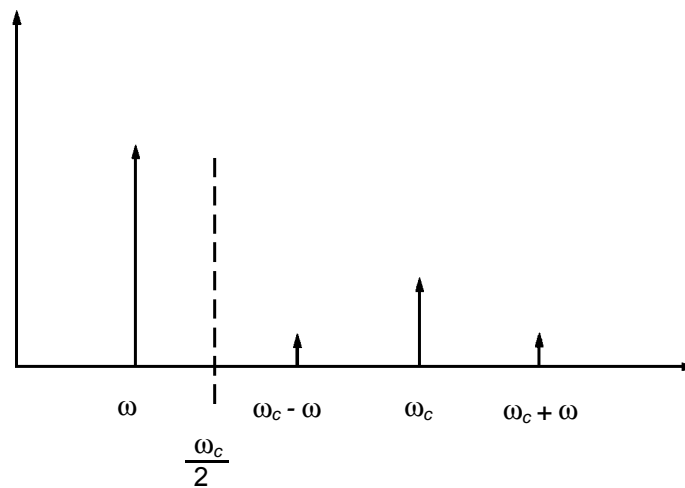

Fig. 19   *Frequency spectrum of an amplitude modulated signal which has been imperfectly demodulated.*

A potential problem with ac modulated systems occurs when the frequency of analysis ω approaches $\omega_c/2$, where ω_c is the carrier frequency. The spectrum of the demodulated signal (Figure 19) will contain contributions at ω, ω_c and $\omega_c \pm \omega$. As ω is increased the fundamental and the lower sideband $\omega_c - \omega$ converge, until at $\omega = \omega_c/2$, they meet and both contribute to the correlator output. Depending on the relative phases of ω and $\omega_c - \omega$ the transfer function analyser at $\omega_c/2$ can take any one of a whole range of values. In practice, ac modulated servos have a frequency response which falls to zero well before $\omega_c/2$, so that this kind of phenomenum does not frequently cause problems.
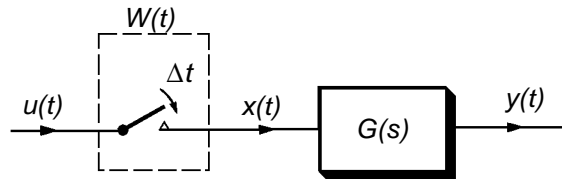
iii) Sampled Systems

Fig. 20   *A system with a sampler.*

The modulation affect described above can cause problems in systems which included sampling action. Specifically, consider a sampled data control system (Figure 20) in which the test signal $u(t)$ is sampled at uniform intervals Δt seconds apart. The sampling action can be written as

$$x(t) = u(t) \; W(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textbf{22.}$$

Where the sampler $W(t)$ is a train of impulses spaced Δt seconds apart. Now the sampler can be considered as a series expansion in terms of harmonies of ω_s, where $\omega_s = 2\pi/\Delta t$.

$$\text{i.e.} \quad W(t) = \frac{1}{\Delta t}\left(1 + 2\sum_{k=1}^{\infty} \cos k\omega_s t\right) \qquad\qquad\qquad \textbf{23.}$$

If the input is a sinewave of frequency ω and relative phase Φ then the sampled output is

$$x(t) = \frac{1}{\Delta t}\left[\sin(\omega t + \Phi) + 2\sum_{k=1}^{\infty} \sin(\omega t + \Phi)\cos k\omega_s t\right] \qquad\qquad \textbf{24.}$$

or

$$x(t) = \frac{1}{\Delta t}\left[\sin(\omega t + \Phi) + \sum_{k=1}^{\infty}\left[\sin\left(k\omega_s t + \omega t + \Phi\right) - \sin\left(k\omega_s t - \omega t - \Phi\right)\right]\right] \qquad \textbf{25.}$$
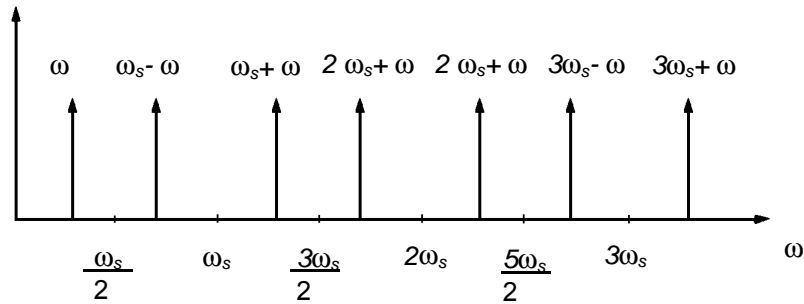

Fig. 21   *Frequency spectrum of the sampled signal x(t).*

The frequency spectrum of this is shown in Figure 21. Note that when $\omega = \omega_s/2$ the lower sideband at $\omega_s - \omega$ coincides with the fundamental frequency ω and gives a component of $x(t)$ at frequency $\omega = \omega_s/2$ demarked $\chi(t)$ and defined by

$$\chi(t) = \frac{1}{\Delta t}\left[\sin\left(\omega t + \Phi\right) - \sin(\omega t - \Phi)\right] \qquad\qquad\qquad \textbf{26.}$$

$$= \frac{2}{\Delta t}\cos\omega t \sin\Phi \qquad\qquad\qquad\qquad\qquad$$